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A Regularity of Dirichlet Problem with the Data belongs to Generalized Morrey Spaces

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Abstract. In this paper, we prove that the weak solution of the Dirichlet problem and its modulus of gradient, with the data belongs to the generalized Morrey spaces, are the element of some weak Morrey spaces.

INTRODUCTION

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^n$, with $n \geq 3$. We are interested to investigate the following Dirichlet problem

$$\begin{cases} Lu = f, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{1}$$

where L is the divergent elliptic operator. Here the function f, which is called data, belongs to the generalized Morrey spaces defined by Nakai [1]. The Morrey spaces played an important role not only in the theory of partial differential equations [2, 3, 4] but also in the theory of function spaces [5, 6, 7, 8, 9].

For the case the data f is the element of the suitable (classical) Morrey spaces, Di Fazio [10] has proved the weak solution of Dirichlet problem (1) belongs to the some weak (classical) Morrey spaces. This result was generalized by Borrello [11], that was used the system of Hörmander vector fields in $\mathbb{R}^n$ to define the degenerate elliptic operator L. Both of [10] and [11] used the Green functions introduced in [12]. In 2020, by using the Green function from [13], Di Fazio [14] obtained similar result as in [10]. Continuing the work of Di Fazio, Tumalun and Tuerah [15] have recently proved that the problem the weak solution gradient of problem (1) belongs to some weak Morrey spaces. Recently, by assuming the elliptic operator L with drift term and the data belongs to some Morrey spaces, Cirmi et al. [16] proved the weak solution of (1) is Hölder continuous and its gradient belongs to some Morrey spaces. For the data belongs to some generalized weighted Morrey spaces, in [17] they obtained that the gradient of the weak solution of problem (1) also belong to the same spaces.

Considering the data f in problem (1) is the element of generalized Morrey spaces, our regularity result in this paper generalizes the previous works which are obtained by previous authors, in the setting L is the divergent elliptic operator. We also show that the modulus of weak solution gradient belongs to some generalized Morrey spaces.

NOTATIONS, FUNCTION SPACES, AND SOME TOOLS

Throughout this paper, let Ω be an open, bounded, and connected subset of $\mathbb{R}^n$, with $n \geq 3$. For every E measurable subset of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$, notation $|E|$ denoted the (Lebesgue) measure of E and $|a|$ denoted the Euclidean norm for a. For $r > 0$, we define

$$B(a,r) = \{y \in \mathbb{R}^n : |y - a| < r\},$$

and

$$\Omega(a,r) = \Omega \cap B(a,r) = \{y \in \Omega : |y - a| < r\}.$$

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The generalized Morrey spaces $L^{1,\varphi}(\Omega)$, for $\varphi:(0,\infty)\longrightarrow(0,\infty)$, is the set of all functions $f\in L^1(\Omega)$ which satisfies

$$\|f\|_{L^{1,\varphi}}=\sup_{x\in\Omega,r>0}\left(\frac{1}{\varphi(r)}\int_{\Omega(a,r)}|f(y)|dy\right)<\infty.$$

Meanwhile, for $1\le p<\infty$, the set of all measurable functions f defined on Ω which satisfies

$$\|f\|_{wL^{p,\varphi}}=\sup_{x\in\Omega,r>0}\left(\frac{\sup_{t>0}t|\{x\in\Omega(a,r):|f(x)|>t\}|^{\frac{1}{p}}}{\varphi(r)^{\frac{1}{p}}}\right)<\infty,$$

is called the generalized weak Morrey spaces and denoted by $wL^{p,\varphi}(\Omega)$.

The function φ above is always assumed to satisfy the following conditions. First, φ is ***n*-almost decreasing** function, that is,

$$s\le t\implies\frac{\varphi(s)}{s^n}\ge C_0\frac{\varphi(t)}{t^n},$$

for a constant positive C_0. Second, φ satisfies **Nakai's condition**, that is, there exist $\alpha<0$ such that, for all $\delta>0$,

$$\int_\delta^\infty\frac{\varphi(t)}{t^{n-1}}dt\le C_1\delta^\alpha,$$

for a constant positive C_1.

For $q=1,2$, let $W^{1,q}(\Omega)$ be the Sobolev spaces. The closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$ under the Sobolev norm is denoted by $W_0^{1,q}(\Omega)$. The notation $H^{-1}(\Omega)$ is the dual space of $W_0^{1,2}(\Omega)$. Now, we consider the following second order divergent elliptic operator

$$Lu=-\sum_{i,j=1}^{\infty}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right),$$

where $u\in W_0^{1,2}(\Omega)$,

$$a_{ij}=a_{ji}\in L^\infty(\Omega),\quad i,j=1,2,\dots,n,$$

and there exists $\lambda>0$ such that,

$$\lambda|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\le\lambda^{-1}|\xi|^2,$$

for every $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ and for almost all $x\in\Omega$. Moreover, we assume a regularity condition of the coefficients of the operator L, that is, $a_{i,j}$ satisfies Dini-continuous condition

$$\left|a_{i,j}(x)-a_{i,j}(y)\right|\le\omega(|x-y|),\quad\forall x,y\in\Omega.$$

Here $\omega:(0,\infty)\longrightarrow(0,\infty)$ is non-decreasing, satisfies

$$\omega(2t)\le C\omega(t)$$

for a constant C and for all $t>0$, and

$$\int_0^\infty\frac{\omega(t)}{t}dt<\infty.$$

In the Dirichlet problem (1), let $f\in L^{1,\varphi}(\Omega)\cap H^{-1}(\Omega)$ and L be the second order divergent elliptic operator above. A function $u\in W_0^{1,2}(\Omega)$ is a weak solution of problem (1) if

$$\int_\Omega\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial\phi(x)}{\partial x_i}dx=\int_\Omega f(x)\,\phi(x)dx,$$

for every $\phi\in C_0^\infty(\Omega)$. Furthermore, a function $u\in L^1(\Omega)$ is a very weak solution of problem (1) if for every $\phi\in W_0^{1,2}(\Omega)\cap C(\overline{\Omega})$ such that $L\phi\in C(\overline{\Omega})$, we have

$$\int_\Omega u(x)\,L\phi(x)dx=\int_\Omega f(x)\,\phi(x)dx.$$

One of important facts, we will use later, that if u is the very weak solution of problem (1), then u is the weak solution of problem (1) (see [12] and [14]).

The following theorem stated the existence of the Green function for the operator L and domain Ω.

Theorem 1. (Grüter and Widman, [13]). *There exists a unique function $G:\Omega\times\Omega\to[0,\infty]$ such that, for each $y\in\Omega$ and $r>0$,*

$$G(\cdot,y) \in W^{1,2}\big(\Omega \backslash B(y,r)\big) \cap W_0^{1,1}(\Omega), \tag{2}$$

and for all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial G(x,y)}{\partial x_i} \frac{\partial \phi(x)}{\partial x_j} dx = \phi(y). \tag{3}$$

Furthermore, there exists a positive constant $C = C(n,\lambda)$ such that,

$$G(x,y) \leq C|x-y|^{2-n}, \tag{4}$$

for all $x, y \in \Omega$, $x \neq y$, and there a positive constant $C = (n,\lambda,\omega,\Omega)$ such that,

$$|\nabla_x G(x,y)| \leq C|x-y|^{1-n}, \tag{5}$$

for all $x, y \in \Omega$, $x \neq y$.

The function G in Theorem 1 is called the **Green function**. Now, fix $y \in \Omega$. According to (2), $G(\cdot,y)$ has a weak derivative in Ω, which is denoted by $\frac{\partial G(x,y)}{\partial x_i}$, for $i = 1, \dots, n$. Therefore

$$\int_\Omega \frac{\partial G(x,y)}{\partial x_i} \phi(x)\, dx = -\int_\Omega G(x,y) \frac{\partial \phi(x)}{\partial x_i} dx, \tag{6}$$

for all $\phi \in C_0^\infty(\Omega)$.

For every $f \in L_{loc}^1(\mathbb{R}^n)$, let M be the Hardy-Littlewood maximal operator, defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $x \in \mathbb{R}^n$. Since φ is n-almost decreasing function, then the following boundedness property of maximal function holds.

Theorem 2. (Nakai, [18]) *Let $a \in \Omega$ and $r > 0$. If $f \in L^{1,\varphi}(\Omega)$, then there exists a positive constant C, which is independent from a and r, such that,*

$$\sup_{t>0} t|\{y \in \Omega(a,r) : |f(y)| > t\}| \leq C\varphi(r)\|f\|_{L^{1,\varphi}}.$$

The notation $C(\alpha, \beta, \dots, \gamma)$, which will be appeared in all proofs in this paper, represents the positive constant which depends on $\alpha, \beta, \dots, \gamma$ and can be vary from line to line in its occurrence.

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RESULT AND DISCUSSION

We start defining an integral operator which will represent the weak solution of Dirichlet problem (1). Let G be the Green function for the operator L and domain Ω. For every $f \in L^{1,\varphi}(\Omega)$, we define

$$u(x) = \int_\Omega G(x,y) f(y) dy, \tag{7}$$

for every $x \in \Omega$.

Theorem 3. *There exists a constant positive C such that,*

$$t^{\frac{\alpha-2}{\alpha}} |\{x \in \Omega(a,r) : |u(x)| > t\}| \leq C\|f\|_{L^{1,\varphi}}^{\frac{\alpha-2}{\alpha}} \varphi(r),$$

for every $a \in \mathbb{R}^n$, $r > 0$, and $t > 0$.

Proof. Let $x \in \Omega$ and $\delta > 0$. We have

$$\int_\Omega \frac{|f(y)|}{|x-y|^{n-2}} dy = \int_{\Omega(y,2\delta)} \frac{|f(y)|}{|x-y|^{n-2}} dy + \int_{\Omega \backslash B(y,2\delta)} \frac{|f(y)|}{|x-y|^{n-2}} dy = I_1 + I_2.$$

The estimation of I_1 is the following

$$I_1 \leq C(n)\delta^2 M(f)(x).$$

By using Nakai's condition, we estimate the I_2 as follows

$$\begin{aligned} I_2 &\leq \sum_{k=1}^{\infty} \int_{2^k\delta \leq |x-y| < 2^{k+1}\delta} \frac{|f(y)|}{|x-y|^{n-2}} dy \leq C(n) \sum_{k=1}^{\infty} \frac{1}{(2^{k+1}\delta)^{n-2}} \frac{\varphi(2^{k+1}\delta)}{\varphi(2^{k+1}\delta)} \int_{2^k\delta \leq |x-y| < 2^{k+1}\delta} |f(y)|\, dy \\ &\leq C(n)\|f\|_{L^{1,\varphi}} \int_\delta^\infty \frac{\varphi(t)}{t^{n-1}} dt \end{aligned}$$

$$\leq C_2\|f\|_{L^{1,\varphi}}\delta^\alpha,$$

where $C_2 = C_2(n, C_1)$. Therefore

$$\int_\Omega \frac{|f(y)|}{|x-y|^{n-2}}dy \leq C_3\left[\delta^2 M(f)(y) + \|f\|_{L^{1,\varphi}}\delta^\alpha\right],$$

where $C_3 = C_3(n, C_2)$. To minimize this inequality, we choose

$$\delta = \left(\frac{M(f)(x)}{\|f\|_{L^{1,\varphi}}}\right)^{\frac{1}{\alpha-2}}$$

to obtain

$$\int_\Omega \frac{|f(y)|}{|x-y|^{n-2}}dy \leq C_3 M(f)(x)^{\frac{\alpha}{\alpha-2}}\|f\|_{L^{1,\varphi}}{}^{\frac{2}{2-\alpha}}.$$

Combining this inequality and (4), we have

$$|u(x)| \leq C_3 M(f)(x)^{\frac{\alpha}{\alpha-2}}\|f\|_{L^{1,\varphi}}{}^{\frac{2}{2-\alpha}}.$$

Let $a \in \Omega$ and $r > 0$ be given. Then

$$|\{x \in \Omega(a,r): |u(x)| > t\}| \leq \left|\left\{x \in \Omega(a,r): M(f)(x) > C_3 t^{\frac{\alpha-2}{\alpha}}\|f\|_{L^{1,\varphi}}{}^{\frac{2}{\alpha}}\right\}\right|,$$

holds for every $t > 0$. The right hand side of this inequality can be bounded by applying Theorem 2, that is,

$$\left|\left\{x \in \Omega(a,r): M(f)(x) > C_3 t^{\frac{\alpha-2}{\alpha}}\|f\|_{L^{1,\varphi}}{}^{\frac{2}{\alpha}}\right\}\right| \leq C\frac{\varphi(r)}{\left(t^{\frac{\alpha-2}{\alpha}}\right)}\|f\|_{L^{1,\varphi}}{}^{\frac{\alpha-2}{\alpha}},$$

where the constant positive C independent from a, r, and t. Whence

$$t^{\frac{\alpha-2}{\alpha}}|\{x \in \Omega(a,r): |u(x)| > t\}| \leq C\|f\|_{L^{1,\varphi}}{}^{\frac{\alpha-2}{\alpha}}\varphi(r).$$

The theorem is proved. ■

The following two lemmas will play important role to compute the weak derivative of u and to show that u is the weak solutions of Dirichlet problem (1).

Lemma 1. $u \in L^1(\Omega)$.

Proof. Let $r > 0$. Then by compactness, there exists a natural number m, which depends on n, and $a_1, a_2, \dots, a_m \in \Omega$ such that, $\Omega \subseteq \bigcup_{k=1}^m B(a_k, r)$. This implies $\Omega \subseteq \bigcup_{k=1}^m \Omega(a_k, r)$. Since Ω is bounded, then $|\Omega(a_k,r)| < \infty$, for every $k = 1, 2, \dots, m$. By the Cavelieri principle and Theorem 3, we have

$$\begin{aligned}\int_{\Omega(a_k,r)}|u(x)|\,dx &= \int_0^\infty |\{x \in \Omega(a_k,r): |u(x)| > t\}|dt \\ &= \int_0^{|\Omega(a_k,r)|} |\{x \in \Omega(a_k,r): |u(x)| > t\}|dt \\ &\quad + \int_{|\Omega(a_k,r)|}^\infty |\{x \in \Omega(a_k,r): |u(x)| > t\}|dt \\ &\leq \int_0^{|\Omega(a_k,r)|}|\Omega(a_k,r)|dt + C\int_{|\Omega(a_k,r)|}^\infty t^{\frac{2-\alpha}{\alpha}}dt \\ &= |\Omega(a_k,r)|^2 + C|\Omega(a_k,r)|^{\frac{2-\alpha}{\alpha}+1}\end{aligned}$$

is finite since $\frac{2-\alpha}{\alpha} + 1 < 0$. Thus

$$\int_\Omega |u(x)|\,dx \leq \sum_{k=1}^m \int_{\Omega(a_k,r)}|u(x)|\,dx \leq \sum_{k=1}^m \left(|\Omega(a_k,r)|^2 + C|\Omega(a_k,r)|^{\frac{2-\alpha}{\alpha}+1}\right) < \infty.$$

We conclude that $u \in L^1(\Omega)$. ■

Lemma 2. *If* $\phi \in C_0^\infty(\Omega)$, *then* $u\frac{\partial\phi(x)}{\partial x_i} \in L^1(\Omega)$.

Proof. The proof immediately follows from the inequality

$$\int_\Omega u(x)\frac{\partial\phi(x)}{\partial x_i}dx \leq \max_{x\in\Omega}\left|\frac{\partial\phi(x)}{\partial x_i}\right|\int_\Omega |u(x)|\,dx < \infty,$$

which uses Lemma 1. ■

Now we will compute the weak derivative of u in the next lemma.

Lemma 3. *The weak derivative of u is given by*

$$\frac{\partial u(x)}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\int_\Omega G(x,y)f(y)dy\right) = \int_\Omega \frac{\partial G(x,y)}{\partial x_i} f(y)dy.$$

Proof. Let ϕ be an arbitrary element of $C_0^\infty(\Omega)$. According to (6), we have

$$\begin{aligned}\int_\Omega \left(\int_\Omega G(x,y)f(y)dy\right)\frac{\partial\phi(x)}{\partial x_i}dx &= \int_\Omega f(y)\left(\int_\Omega G(x,y)\frac{\partial\phi(x)}{\partial x_i}dx\right)dy \\ &= -\int_\Omega f(y)\left(\int_\Omega \frac{\partial G(x,y)}{\partial x_i}\phi(x)\,dx\right)dy \\ &= -\int_\Omega \left(\int_\Omega \frac{\partial G(x,y)}{\partial x_i}f(y)\,dy\right)\phi(x)dx.\end{aligned}$$

Here we use Fubini theorem which is guaranteed by Lemma 2 ■

According to Lemma 1, it is enough to prove that u is the very weak solution of problem (1).

Theorem 4. *The function u is the weak solution of Dirichlet problem (1) and belongs to $wL^{\frac{\alpha-2}{\alpha},\varphi}(\Omega)$.*

Proof. Let ϕ be an arbitrary element of $H_0^1(\Omega)\cap C(\overline{\Omega})$ such that $L\phi \in C(\overline{\Omega})$. By using the Green function property (3) and Lemma 3, we have

$$\begin{aligned}\int_\Omega \phi(y)f(y)dy &= \int_\Omega \left(\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial G(x,y)}{\partial x_i}\frac{\partial\phi(x)}{\partial x_j}dx\right)f(y)dy \\ &= \int_\Omega \sum_{i,j=1}^n a_{ij}(x)\left(\int_\Omega \frac{\partial G(x,y)}{\partial x_i}f(y)dy\right)\frac{\partial\phi(x)}{\partial x_j}dx = \int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial\phi(x)}{\partial x_j}d \\ &= \int_\Omega u(x)L\phi(x)dx.\end{aligned}$$

We also apply Fubini theorem in this calculation since $\phi f \in L^1(\Omega)$. This means that u is the very weak solution of problem (1). Now, by virtue to Theorem 3 we see that $u \in wL^{\frac{\alpha-2}{\alpha},\varphi}(\Omega)$. ■

The last theorem below shows that modulus the gradient of the weak solution problem (1) belongs to the weak Morrey spaces $wL^{\frac{\alpha-2}{\alpha-1},\varphi}(\Omega)$.

Theorem 5. $|\nabla u| \in wL^{\frac{\alpha-2}{\alpha-1},\varphi}(\Omega)$.

Proof. The proof follows the same method as in Theorem 3. Let $x\in\Omega$ and $\delta > 0$. We have

$$\int_\Omega \frac{|f(y)|}{|x-y|^{n-1}}dy \le CM(f)(x)^{\frac{\alpha-1}{\alpha-2}}\|f\|_{L^{1,\varphi}}^{\frac{1}{2-\alpha}}.$$

Combining this inequality, inequality (5), and Lemma 3, we have

$$|\nabla u(x)| \le CM(f)(x)^{\frac{\alpha-1}{\alpha-2}}\|f\|_{L^{1,\varphi}}^{\frac{1}{2-\alpha}}.$$

By applying Theorem 2, then

$$t^{\frac{\alpha-2}{\alpha-1}}|\{x\in\Omega(a,r): |\nabla u(x)| > t\}| \le C\|f\|_{L^{1,\varphi}}^{\frac{1}{\alpha-1}}\varphi(r).$$

Here the positive constant C is independent from a, r, and t. The theorem is already proved.■

CONCLUSION

The weak solution of the Dirichlet problem (1), by assuming the data is an element of some generalized Morrey spaces, belongs to some weak generalized Morrey spaces. Furthermore, this weak solution gradient also belongs to some weak generalized Morrey spaces.

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