

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/368282673>

# Proper Inclusion Between Vanishing Morrey Spaces and Morrey Spaces

Article · April 2021

---

CITATION

1

READS

6

1 author:



[Nicky Kurnia Tumulun](#)

Universitas Negeri Manado

10 PUBLICATIONS 23 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:



Konsep Sudut pada Ruang Bernorma- $n$  dan Aplikasi Kelas Stummel dalam Teori Regularitas [View project](#)

# Proper Inclusion Between Vanishing Morrey Spaces and Morrey Spaces

Nicky K. Tumulun<sup>1\*</sup>

<sup>1</sup>Mathematics Department, Faculty of Mathematics and Natural Sciences, Universitas Negeri Manado, Tondano Selatan, Minahasa, Sulawesi Utara, Indonesia, KP 95681.

Email: [nickytumalun@unima.ac.id](mailto:nickytumalun@unima.ac.id)

Manuscript submitted : February 2021

Accepted for publication : April 2021.

---

**Abstract:** In this paper, I give an explicit function which belongs to the Morrey spaces but not in the vanishing Morrey spaces. Therefore, I obtain that the Morrey spaces contain the vanishing Morrey spaces properly.

2010 Mathematical Subject Classification: 46E30, 42B35.

**Keywords:** Morrey spaces, vanishing Morrey spaces.

---

## 1. Introduction and Statement of The Main Result

Let  $1 \leq p < \infty$  and  $0 < \lambda < n$ . A function  $f \in L_{loc}^p(\mathbb{R}^n)$  is said to be an element of the **Morrey spaces**  $L^{p,\lambda}(\mathbb{R}^n)$  if

$$\|f\|_{L^{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \left( r^{-\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

For more information regarding to the Morrey spaces  $L^{p,\lambda}(\mathbb{R}^n)$ , see [1]. Now, for every  $f \in L^{p,\lambda}(\mathbb{R}^n)$ , we set

$$\mathcal{M}_f(r) = \sup_{x \in \mathbb{R}^n} \left( r^{-\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

The function  $f$  belongs to the **vanishing Morrey spaces**  $VL^{p,\lambda}(\mathbb{R}^n)$  if

$$\lim_{r \rightarrow 0} \mathcal{M}_f(r) = 0.$$

The vanishing Morrey spaces were introduced in [2] and have some applications in elliptic partial differential equations, operator theory, and approximation in Morrey spaces [3, 4, 5].

It is clear from the definitions above that  $VL^{p,\lambda}(\mathbb{R}^n)$  is a subset of  $L^{p,\lambda}(\mathbb{R}^n)$ . In [4], it stated that this inclusion is proper without giving an explicit function which belongs to  $L^{p,\lambda}(\mathbb{R}^n)$  but not in  $VL^{p,\lambda}(\mathbb{R}^n)$ . In this paper, by using the idea of the proper inclusion between the bounded Stummel modulus classes and the Stummel classes [6] (see also to be published [7] and [8]), we will give an example of that function. Although there are some inclusion relations between the Morrey spaces and the Stummel classes for some appropriate parameters [6, 9], but in general these relations may fail (submitted for publication [10]).

Our main result is stated in the following theorem.

**Theorem 1.** *The vanishing Morrey spaces contain the Morrey spaces properly.*

The proof of this theorem will be given in the next section.

## 2. Proof Of The Main Result

To prove the Theorem 1, we need to give an explicit function belongs to the Morrey spaces but not in the vanishing Morrey spaces. The positive constant  $C = C(n, \lambda)$ , which means depending only on  $n$  and  $\lambda$ , appears in this paper may be vary from line to line.

Let  $n$  be an integer number such that  $n \geq 2$ ,  $0 < \lambda < n$ , and  $1 \leq p < \infty$ . For every integer  $k \geq 3$ , setting  $x_k = (2^{-k}, 0, \dots, 0) \in \mathbb{R}^n$  and

$$f_k(y) = \begin{cases} 8^{(n-\lambda)k} & : y \in B(x_k, 8^{-k}) \\ 0 & : y \notin B(x_k, 8^{-k}). \end{cases}$$

We define a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  which its formula is given by

$$f(y) = \left( \sum_{k=3}^{\infty} f_k(y) \right)^{\frac{1}{p}}. \tag{1}$$

We claim that this function is an element of the Morrey spaces and is not an element of the vanishing Morrey spaces. We need the following lemmas to prove this claim. In the rest of this paper, the notation of the function  $f$  defined in (1).

**Lemma 1.**  $f \in L^p_{loc}(\mathbb{R}^n)$ .

**Proof.** Let  $x \in \mathbb{R}^n$  and  $r > 0$ . Note that

$$\int_{B(x,r)} |f(y)|^p dy = \sum_{k=3}^{\infty} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) dy \leq \sum_{k=3}^{\infty} 8^{(n-\lambda)k} \int_{B(x_k, 8^{-k})} 1 dy = C(n) \sum_{k=3}^{\infty} 8^{-\lambda k} < \infty.$$

The lemma is proved. ■

**Lemma 2.** Let  $k \geq 3$  be an integer,  $x \in \mathbb{R}^n$ , and  $r > 0$ . If  $|x - x_k| < 2(4^{-k})$ , then

$$r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) dy \leq C(n, \lambda).$$

**Proof.** We have two cases: (i)  $2(8^{-k}) \leq |x - x_k| < 2(4^{-k})$  or (ii)  $|x - x_k| < 2(8^{-k})$ . For the case (i), we obtain

$$2(8^{-k}) \leq |x - x_k| < |x - y| + |y - x_k| < r + 8^{-k} \implies r^{-\lambda} < 8^{\lambda k}$$

for every  $y \in B(x, r) \cap B(x_k, 8^{-k})$ . Hence

$$r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) dy = r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} 8^{(n-\lambda)k} dy \leq C(n) 8^{\lambda k} 8^{(n-\lambda)k} 8^{-nk} = C(n).$$

For the case (ii), it is easy to show that  $B(x_k, 8^{-k}) \subseteq B(x, 3(8^{-k}))$ , which implies

$$\begin{aligned} r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) dy &= r^{-\lambda} \int_{B(x,r) \cap B(x_k, 8^{-k})} 8^{(n-\lambda)k} dy \leq 8^{(n-\lambda)k} \int_{B(x,r) \cap B(x_k, 8^{-k})} |x - y|^{-\lambda} dy \\ &\leq 8^{(n-\lambda)k} \int_{B(x, 3(8^{-k}))} |x - y|^{-\lambda} dy = C(n, \lambda) 8^{(n-\lambda)k} 8^{(n-\lambda)(-k)} = C(n, \lambda). \end{aligned}$$

Combining the two cases results above, the lemma is proved. ■

Using the Lemma 1 and the Lemma 2, we will show that the function  $f$  belongs to the Morrey spaces  $L^{p,\lambda}(\mathbb{R}^n)$ .

**Lemma 3.**  $f \in L^{p,\lambda}(\mathbb{R}^n)$ .

**Proof.** From Lemma 1,  $f \in L^p_{loc}(\mathbb{R}^n)$ . Now, let  $x \in \mathbb{R}^n$  and  $r > 0$ . There are two cases: (i)  $x \notin B(x_k, 2(4^{-k}))$  for every  $k \geq 3$ , or (ii)  $x \in B(x_j, 2(4^{-j}))$  for some  $j \geq 3$ . Assume the case (i) holds. For every  $y \in B(x, r) \cap B(x_k, 8^{-k})$ , we have

$$2(4^{-k}) \leq |x - x_k| \leq |x - y| + |y - x_k| < r + 8^{-k} < r + 4^{-k}.$$

This means  $r^{-\lambda} \leq 4^{\lambda k}$  and

$$\begin{aligned} r^{-\lambda} \int_{B(x,r)} |f(y)|^p dy &= \sum_{k=3}^{\infty} r^{-\lambda} 8^{(n-\lambda)k} \int_{B(x,r) \cap B(x_k, 8^{-k})} 1 dy \\ &\leq C(n) \sum_{k=3}^{\infty} 4^{\lambda k} 8^{(n-\lambda)k} 8^{-nk} = C(n) \sum_{k=3}^{\infty} 2^{-\lambda k} < \infty. \end{aligned} \tag{2}$$

Assume the case (ii) holds. Then there is only one  $j \geq 3$  such that  $x \in B(x_j, 2(4^{-j}))$  and since  $\{B(x_k, 2(4^{-k}))\}_{k \geq 3}$  is a disjoint collection. We also obtain  $x \notin B(x_k, 2(4^{-k}))$  for every  $k \geq 3$  with  $k \neq j$ .

By virtue to Lemma 2 and the calculation of (2), we conclude

$$\begin{aligned} r^{-\lambda} \int_{B(x,r)} |f(y)|^p dy &= r^{-\lambda} \sum_{k=3}^{\infty} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) dy \\ &= r^{-\lambda} \int_{B(x,r) \cap B(x_j, 8^{-j})} f_j(y) dy + r^{-\lambda} \sum_{\substack{k=3 \\ k \neq j}}^{\infty} \int_{B(x,r) \cap B(x_k, 8^{-k})} f_k(y) dy \\ &\leq C(n, \lambda) + C(n) \sum_{\substack{k=3 \\ k \neq j}}^{\infty} 2^{-\lambda k} < \infty. \end{aligned} \tag{3}$$

Therefore,

$$\|f\|_{L^{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \left( r^{-\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} \leq C(n, \lambda, p) < \infty,$$

in view of (2) and (3). This completes the proof. ■

**Lemma 4.**  $f \notin VL^{p,\lambda}(\mathbb{R}^n)$ .

**Proof.** Let  $x \in \mathbb{R}^n$  and  $0 < r < 1$ . Choose an integer  $k$  such that  $8^{-k} < r$ . Then

$$\begin{aligned} (\mathcal{M}_f(r))^p &\geq r^{-\lambda} \int_{B(x,r)} |f(y)|^p dy \geq \int_{B(x_k, 8^{-k})} f_k(y) dy \geq \int_{B(x_k, 8^{-k})} 8^{(n-\lambda)k} dy \\ &\geq C(n) 8^{nk} 8^{-nk} = C(n) > 0. \end{aligned}$$

Hence  $\mathcal{M}_f(r)$  bounded away from zero as  $r$  tends to zero. ■

**Proof of The Theorem 1.** Let  $f$  be defined by (1). According to the Lemma 3 and the Lemma 4, we have  $f \in L^{p,\lambda}(\mathbb{R}^n) \setminus VL^{p,\lambda}(\mathbb{R}^n)$ . Hence  $VL^{p,\lambda}(\mathbb{R}^n)$  is a proper subset of  $L^{p,\lambda}(\mathbb{R}^n)$ . ■

### Acknowledgment

The author thanks Dr. Harmanus Batkunde for the useful discussion and the anonymous referee for the useful comments.

## References

- [1] Sawano, Y., Di Fazio, G. and Hakim, D., *Morrey Spaces (2020) Introduction and Application to Integral Operator and PDE's, Volume I & II*, Boca Raton: CRC Press,.
- [2] Vitanza, C. (2020). Functions with vanishing Morrey norm and elliptic partial differential equations, *Proceedings of Methods of Real Analysis and Partial Differential Equations*, Capri.
- [3] Ragusa, M.,(2008). Commutators of fractional integral operators on vanishing Morrey spaces, *J. Global Optim.*, 40(1). 361-368.
- [4] Samko, N., (2013). Maximal, potential and singular operators in vanishing generalized Morrey spaces, *J. Global. Optim.*, 57, 1385-1399.
- [5] Almeida, A. and Samko, S. (2017). Approximation in Morrey spaces. *J. Funct. Anal.*, 50. 201-230.
- [6] Tumulun, N. K., Hakim, D. I. and Gunawan, H., (2020). Inclusion between generalized Stummel classes and other function spaces. *Math. Inequal. Appl.*, 23(2). 547-562.
- [7] Tumulun, N. K., (2020). Fungsi elemen kelas Stummel modulus terbatas tetapi bukan elemen dari kelas Stummel. *Barekeng*, 14(1).
- [8] Tumulun, N. K. and Tuerah, P. E. A. (2021) Some notes on the inclusion between Morrey spaces, *Submitted For Publication*.
- [9] Tumulun, N. K. and Gunawan, H. (2019). Morrey spaces are embedded between weak Morrey spaces and Stummel classes, *J. Indones. Math. Soc.*, 25(3). 203-209.
- [10] Tumulun, N. K., Hakim, D. I. and Gunawan, H. (2020). Fefferman's inequality and applications in elliptic partial differential equations, arXiv:1810.08952v1 [math.AP].